

# GRÜSS TYPE DISCRETE INEQUALITIES IN INNER PRODUCT SPACES, REVISITED

SEVER S. DRAGOMIR

**ABSTRACT.** Some sharp inequalities of Grüss type for sequences of vectors in real or complex inner product spaces are obtained. Applications for Jensen's inequality for convex functions defined on such spaces are also provided.

## 1. INTRODUCTION

The following inequality of Grüss type for sequences of vectors in inner product spaces has been established in [1].

**Theorem 1.** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ),  $\bar{\mathbf{x}} = (x_1, \dots, x_n) \in H^n$ ,  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ ,  $\bar{\mathbf{p}} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$  with  $\sum_{i=1}^n p_i = 1$ . If  $a, A \in \mathbb{K}$  and  $x, X \in H$  are such that*

$$(1.1) \quad \operatorname{Re}[(A - \alpha_i)(\bar{\alpha}_i - \bar{a})] \geq 0 \quad \text{and} \quad \operatorname{Re}\langle X - x_i, x_i - x \rangle \geq 0$$

*for each  $i \in \{1, \dots, n\}$ ,*

*then we have the inequality*

$$(1.2) \quad 0 \leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{4} |A - a| \|X - x\|.$$

*The constant  $\frac{1}{4}$  is the best possible one in the sense that it cannot be replaced by a smaller constant.*

Another result of this type is embodied in the following theorem that has been obtained in [2].

**Theorem 2.** *Let  $H, \mathbb{K}$  be as above  $\bar{\mathbf{x}} = (x_1, \dots, x_n), \bar{\mathbf{y}} = (y_1, \dots, y_n) \in H^n$  and  $\bar{\mathbf{p}}$  a probability sequence. If  $x, X, y, Y \in H$  are such that*

$$(1.3) \quad \operatorname{Re}\langle X - x_i, x_i - x \rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle Y - y_i, y_i - y \rangle \geq 0$$

*for each  $i \in \{1, \dots, n\}$ ,*

*then we have the inequality*

$$(1.4) \quad 0 \leq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|.$$

*The constant  $\frac{1}{4}$  is best possible in the above sense.*

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On choosing  $x_i = y_i$  ( $i = 1, \dots, n$ ) in Theorem 2, one may obtain the following counterpart of Cauchy-Bunyakovsky-Schwarz inequality

$$(1.5) \quad 0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \frac{1}{4} \|X - x\|^2,$$

provided  $\bar{x}$  and  $\bar{p}$  satisfy the assumptions of Theorem 2.

In the recent paper [3], the author has obtained the following Grüss type inequality for forward difference as well.

**Theorem 3.** *Let  $\bar{x} = (x_1, \dots, x_n)$ ,  $\bar{y} = (y_1, \dots, y_n) \in H^n$  and  $\bar{p} \in \mathbb{R}_+^n$  be a probability sequence. Then one has the inequalities*

$$(1.6) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta x_k\| \max_{k=1, \dots, n-1} \|\Delta y_k\|; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[ \sum_{i=1}^n p_i (1-p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases}$$

The constants 1, 1 and  $\frac{1}{2}$  in the right hand side of the inequality (1.6) are best in the sense that they cannot be replaced by smaller constants.

If one chooses  $p_i = \frac{1}{n}$  ( $i = 1, \dots, n$ ) in (1.6), then the following unweighted inequalities would hold:

$$(1.7) \quad \left| \frac{1}{n} \sum_{i=1}^n \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n y_i \right\rangle \right| \leq \begin{cases} \frac{n^2-1}{12} \max_{k=1, \dots, n-1} \|\Delta x_k\| \max_{k=1, \dots, n-1} \|\Delta y_k\|; \\ \frac{n^2-1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases}$$

Here, the constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are also best possible in the above sense.

The following counterpart inequality of the Cauchy-Bunyakovsky-Schwarz inequality for sequences of vectors in inner product spaces holds.

**Corollary 1.** *With the assumptions in Theorem 3 for  $\bar{x}$  and  $\bar{p}$  one has the inequalities*

$$(1.8) \quad 0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, n-1} \|\Delta x_k\|^2; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[ \sum_{i=1}^n p_i (1-p_i) \right] \left( \sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2. \end{cases}$$

The constants 1, 1 and  $\frac{1}{2}$  are best possible in the above sense.

The following particular inequalities that may be deduced from (1.8) on choosing the equal weights  $p_i = \frac{1}{n}$ ,  $i = 1, \dots, n$  are also of interest

$$(1.9) \quad 0 \leq \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^n x_i \right\|^2 \leq \begin{cases} \frac{n^2-1}{12} \max_{k=1, n-1} \|\Delta x_k\|^2; \\ \frac{n^2-1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2. \end{cases}$$

Here the constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are also best possible.

It is the main aim of this paper to point out a different class of Grüss type inequalities for sequences of vectors in inner product spaces and to apply them for obtaining a reverse of Jenssen's inequality for convex functions defined on such spaces.

## 2. SOME GRÜSS TYPE INEQUALITIES

The following lemma holds (see also [4]).

**Lemma 1.** *Let  $a, x, A$  be vectors in the inner product space  $(H; \langle \cdot, \cdot \rangle)$  over the real or complex number field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) with  $a \neq A$ . The following statements are equivalent:*

- (i)  $\operatorname{Re} \langle A - x, x - a \rangle \geq 0$ ;
- (ii)  $\left\| x - \frac{a+A}{2} \right\| \leq \frac{1}{2} \|A - a\|$ .

*Proof.* For the sake of completeness, we give a simple proof as follows.

Let

$$I_1 := \operatorname{Re} \langle A - x, x - a \rangle = -\operatorname{Re} \langle A, a \rangle - \|x\|^2 + \operatorname{Re} [\langle \overline{x}, A \rangle + \langle x, a \rangle]$$

and

$$\begin{aligned} I_2 &:= \frac{1}{4} \|A - a\|^2 - \left\| x - \frac{a + A}{2} \right\|^2 \\ &= \frac{1}{4} \left( \|A\|^2 - 2 \operatorname{Re} \langle A, a \rangle + \|a\|^2 \right) - \left( \|x\|^2 - \operatorname{Re} \langle x, a + A \rangle + \frac{1}{4} \|A - a\|^2 \right) \\ &= -\operatorname{Re} \langle A, a \rangle - \|x\|^2 + \operatorname{Re} [\langle x, a \rangle + \langle x, A \rangle]. \end{aligned}$$

Since  $\operatorname{Re} \langle x, A \rangle = \left[ \operatorname{Re} \langle \overline{x}, A \rangle \right]$ , we deduce that  $I_1 = I_2$ , showing the desired equivalence. ■

**Remark 1.** If  $H = \mathbb{C}$ ,  $\|\cdot\| = |\cdot|$ , then the following sentences are equivalent

- (a)  $\operatorname{Re} [(A - x)(\overline{x} - \overline{a})] \geq 0$ ;
- (aa)  $\left| x - \frac{a + A}{2} \right| \leq \frac{1}{2} |A - a|$ ,

where  $a, A, x \in \mathbb{C}$ .

If  $H = \mathbb{R}$ ,  $\|\cdot\| = |\cdot|$ , and  $A > a$ , then the following sentences are obviously equivalent:

- (b)  $a \leq x \leq A$ ;
- (bb)  $\left| x - \frac{a + A}{2} \right| \leq \frac{1}{2} |A - a|$ .

The following inequality of Grüss type for sequences of vectors in inner product spaces holds.

**Theorem 4.** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ), and  $\overline{\mathbf{x}} = (x_1, \dots, x_n)$ ,  $\overline{\mathbf{y}} = (y_1, \dots, y_n) \in H^n$ ,  $\overline{\mathbf{p}} \in \mathbb{R}_+^n$  with  $\sum_{i=1}^n p_i = 1$ . If  $x, X \in H$  are such that

$$(2.1) \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for each } i \in \{1, \dots, n\},$$

or, equivalently,

$$(2.2) \quad \left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{for each } i \in \{1, \dots, n\},$$

then one has the inequality

$$\begin{aligned} (2.3) \quad & \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ & \leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\ & \leq \frac{1}{2} \|X - x\| \left[ \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

The constant  $\frac{1}{2}$  is best possible in the first and second inequality in the sense that it cannot be replaced by a smaller constant.

*Proof.* It is easy to see that the following identity holds true

$$(2.4) \quad \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle = \sum_{i=1}^n p_i \left\langle x_i - \frac{x+X}{2}, y_i - \sum_{j=1}^n p_j y_j \right\rangle.$$

Taking the modulus in (2.4) and using the Schwarz inequality in the inner product space  $(H; \langle \cdot, \cdot \rangle)$ , we have

$$\begin{aligned} \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| &\leq \sum_{i=1}^n p_i \left| \left\langle x_i - \frac{x+X}{2}, y_i - \sum_{j=1}^n p_j y_j \right\rangle \right| \\ &\leq \sum_{i=1}^n p_i \left\| x_i - \frac{x+X}{2} \right\| \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|, \end{aligned}$$

and the first inequality in (2.3) is proved.

Using the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences and the calculation rules in inner product spaces, we have

$$\sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \leq \left[ \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|^2 \right]^{\frac{1}{2}}$$

and

$$\sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|^2 = \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2$$

giving the second part of (2.3).

To prove the sharpness of the constant  $\frac{1}{2}$  in the first inequality in (2.3), let us assume that, under the assumptions of the theorem, the inequality holds with a constant  $C > 0$ , i.e.,

$$(2.5) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq C \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|.$$

Consider  $n = 2$  and observe that

$$\begin{aligned} \sum_{i=1}^2 p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^2 p_i x_i, \sum_{i=1}^2 p_i y_i \right\rangle &= p_2 p_1 \langle x_2 - x_1, y_2 - y_1 \rangle, \\ \sum_{i=1}^2 p_i \left\| y_i - \sum_{j=1}^2 p_j y_j \right\| &= 2p_2 p_1 \|y_2 - y_1\| \end{aligned}$$

and then, by (2.5), we deduce

$$(2.6) \quad p_2 p_1 |\langle x_2 - x_1, y_2 - y_1 \rangle| \leq 2C \|X - x\| p_2 p_1 \|y_2 - y_1\|.$$

If we choose  $p_1, p_2 > 0$ ,  $y_2 = x_2$ ,  $y_1 = x_1$  and  $x_2 = X$ ,  $x_1 = x$  with  $x \neq X$ , then (2.2) holds and from (2.6) we deduce  $C \geq \frac{1}{2}$ .

The fact that  $\frac{1}{2}$  is best possible in the second inequality may be proven in a similar manner and we omit the details. ■

**Remark 2.** If  $\bar{x}$  and  $\bar{y}$  satisfy the assumptions of Theorem 2, or equivalently

$$(2.7) \quad \left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \|X - x\|, \quad \left\| y_i - \frac{y+Y}{2} \right\| \leq \frac{1}{2} \|Y - y\|,$$

for each  $i \in \{1, \dots, n\}$ , then by Theorem 4 we may state the following sequence of inequalities improving the Grüss inequality (2.4)

$$(2.8) \quad \begin{aligned} 0 &\leq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ &\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}} \\ (2.9) \quad &\leq \frac{1}{4} \|X - x\| \|Y - y\|. \end{aligned}$$

In particular, for  $x_i = y_i$  ( $i = 1, \dots, n$ ), one has

$$(2.10) \quad 0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| x_i - \sum_{j=1}^n p_j x_j \right\|$$

and the constant  $\frac{1}{2}$  is best possible.

The following result is connected to Theorem 1 from Introduction.

**Theorem 5.** Let  $(H; \langle \cdot, \cdot \rangle)$  and  $\mathbb{K}$  be as above and  $\bar{x} = (x_1, \dots, x_n) \in H^n$ ,  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$  and  $\bar{p}$  a probability vector. If  $x, X \in H$  are such that (2.1) or, equivalently, (2.2) holds, then we have the inequality

$$(2.11) \quad \begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\ &\leq \frac{1}{2} \|X - x\| \left[ \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

The constant  $\frac{1}{2}$  in the first and second inequalities is best possible in the sense that it cannot be replaced by a smaller constant.

*Proof.* We start with the following equality that may be easily verified by direct calculation

$$(2.12) \quad \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i \left( \alpha_i - \sum_{j=1}^n p_j \alpha_j \right) \left( x_i - \frac{x+X}{2} \right).$$

If we take the norm in (2.12), we deduce

$$\begin{aligned}
\left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| &\leq \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \left\| x_i - \frac{x + X}{2} \right\| \\
&\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\
&\leq \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i \left( \alpha_i - \sum_{j=1}^n p_j \alpha_j \right)^2 \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

proving the inequality (2.11).

The fact that the constant  $\frac{1}{2}$  is sharp may be proven in a similar manner to the one embodied in the proof of Theorem 4. We omit the details. ■

**Remark 3.** If  $\bar{x}$  and  $\bar{\alpha}$  satisfy the assumption of Theorem 2, or, equivalently,

$$\left\| \alpha_i - \frac{a + A}{2} \right\| \leq \frac{1}{2} |A - a|, \quad \left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\|,$$

for each  $i \in \{1, \dots, n\}$ , then by Theorem 5 we may state the following sequence of inequalities improving the Grüss inequality (1.2),

$$\begin{aligned}
(2.13) \quad 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\
&\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\
&\leq \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{4} |A - a| \|X - x\|.
\end{aligned}$$

**Remark 4.** If in (2.11) we choose  $x_i = \alpha_i \in \mathbb{C}$  and assume that  $|\alpha_i - \frac{a+A}{2}| \leq \frac{1}{2}|A-a|$ , where  $a, A \in \mathbb{C}$ , then we get the following interesting inequality for complex numbers

$$\begin{aligned} 0 &\leq \left| \sum_{i=1}^n p_i \alpha_i^2 - \left( \sum_{i=1}^n p_i \alpha_i \right)^2 \right| \\ &\leq \frac{1}{2} |A-a| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\ &\leq \frac{1}{2} |A-a| \left[ \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

### 3. APPLICATIONS FOR CONVEX FUNCTIONS

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real inner product space and  $F : H \rightarrow \mathbb{R}$  a Fréchet differentiable convex function on  $H$ . If  $\nabla F : H \rightarrow H$  denotes the gradient operator associated to  $F$ , then we have the inequality

$$(3.1) \quad F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle$$

for each  $x, y \in H$ .

The following result holds.

**Theorem 6.** Let  $F : H \rightarrow \mathbb{R}$  be as above and  $z_i \in H$ ,  $i \in \{1, \dots, n\}$ . Suppose that there exists the vectors  $m, M \in H$  such that either

$$(3.2) \quad \langle \nabla F(z_i) - m, M - \nabla F(z_i) \rangle \geq 0 \quad \text{for each } i \in \{1, \dots, n\};$$

or, equivalently,

$$(3.3) \quad \left\| \nabla F(z_i) - \frac{m+M}{2} \right\| \leq \frac{1}{2} \|M - m\| \quad \text{for each } i \in \{1, \dots, n\}.$$

If  $q_i \geq 0$  ( $i \in \{1, \dots, n\}$ ) with  $Q_n := \sum_{i=1}^n q_i > 0$ , then we have the following converse of Jensen's inequality

$$\begin{aligned} (3.4) \quad 0 &\leq \frac{1}{Q_n} \sum_{i=1}^n q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^n q_i z_i\right) \\ &\leq \frac{1}{2} \|M - m\| \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i - \frac{1}{Q_n} \sum_{j=1}^n q_j z_j \right\| \\ &\leq \frac{1}{2} \|M - m\| \left[ \frac{1}{Q_n} \sum_{i=1}^n q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned}$$



*Proof.* We know, see for example [2, Eq. (4.4)], the following counterpart of Jensen's inequality for Fréchet differentiable convex functions

$$(3.5) \quad 0 \leq \frac{1}{Q_n} \sum_{i=1}^n q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^n q_i z_i\right) \\ \leq \frac{1}{Q_n} \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \frac{1}{Q_n} \sum_{i=1}^n q_i \nabla F(z_i), \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\rangle$$

holds.

Now, if we use Theorem 4 for the choices  $x_i = \nabla F(z_i)$ ,  $y_i = z_i$  and  $p_i = \frac{1}{Q_n} q_i$ ,  $i \in \{1, \dots, n\}$ , then we can state the inequality

$$(3.6) \quad \frac{1}{Q_n} \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \frac{1}{Q_n} \sum_{i=1}^n q_i \nabla F(z_i), \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\rangle \\ \leq \frac{1}{2} \|M - m\| \frac{1}{Q_n} \sum_{i=1}^n q_i \left\| z_i - \frac{1}{Q_n} \sum_{j=1}^n q_j z_j \right\| \\ \leq \frac{1}{2} \|M - m\| \left[ \frac{1}{Q_n} \sum_{i=1}^n q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\|^2 \right]^{\frac{1}{2}}.$$

Utilizing (3.5) and (3.6), we deduce the desired result (3.4). ■

If more information is available about the vector sequence  $\bar{z} = (z_1, \dots, z_n) \in H^n$ , then we may state the following corollary.

**Corollary 2.** *With the assumptions in Theorem 6 and if there exists the vectors  $z, Z \in H$  such that either*

$$(3.7) \quad \langle z_i - z, Z - z_i \rangle \geq 0 \quad \text{for each } i \in \{1, \dots, n\};$$

*or, equivalently*

$$(3.8) \quad \left\| z_i - \frac{z + Z}{2} \right\| \leq \frac{1}{2} \|Z - z\| \quad \text{for each } i \in \{1, \dots, n\},$$

*then we have the inequality*

$$(3.9) \quad 0 \leq \frac{1}{Q_n} \sum_{i=1}^n q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^n q_i z_i\right) \\ \leq \frac{1}{2} \|M - m\| \frac{1}{Q_n} \sum_{i=1}^n q_i \left\| z_i - \frac{1}{Q_n} \sum_{j=1}^n q_j z_j \right\| \\ \leq \frac{1}{2} \|M - m\| \left[ \frac{1}{Q_n} \sum_{i=1}^n q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\|^2 \right]^{\frac{1}{2}} \\ \leq \frac{1}{4} \|M - m\| \|Z - z\|.$$

**Remark 5.** *Note that the inequality between the first term and the last term in (3.9) was firstly proved in [2, Theorem 4.1]. Consequently, the above corollary provides an improvement of the reverse of Jensen's inequality established in [2].*

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SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY,  
PO Box 14428, MCMC 8001, VICTORIA, AUSTRALIA.

*E-mail address:* `sever@matilda.vu.edu.au`

*URL:* `http://rgmia.vu.edu.au/SSDragomirWeb.html`